ΓΕΝΙΚΕΣ ΕΙΣΙΤΗΡΙΕΣ ΕΞΕΤΑΣΕΙΣ 1972
ΑΠΑΝΤΗΣΕΙΣ ΣΤΗΝ ΤΡΙΓΩΝΟΜΕΤΡΙΑ
(ΠΟΛΥΤΕΧΝΙΚΟΣ ΚΥΚΛΟΣ)
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Ζήτημα 1ο

$$\sigma\upsilon\nu^4\alpha = \frac{3}{8} + \frac{1}{2}\sigma\upsilon\nu 2\alpha + \frac{1}{8}\sigma\upsilon\nu 4\alpha \quad (1)$$

$$\sigma\upsilon\nu\alpha + \sigma\upsilon\nu(\alpha+\omega) + \sigma\upsilon\nu(\alpha+2\omega) = \frac{\sigma\upsilon\nu(\alpha+\omega)\cdot\eta\mu\frac{3\omega}{2}}{\eta\mu\frac{\omega}{2}} \quad (2)$$

$$\eta\mu 3\alpha = 3\eta\mu\alpha - 4\eta\mu^3\alpha \Leftrightarrow \eta\mu 3\alpha = \eta\mu\alpha\cdot(3-4\eta\mu^2\alpha) \quad (3)$$

$$S = \sigma\upsilon\nu^4(-3^0) + \sigma\upsilon\nu^4 15^0 + \sigma\upsilon\nu^4 33^0$$

$$\overset{(1)}{=} \frac{3}{8}+\frac{1}{2}\sigma\upsilon\nu(-6^0)+\frac{1}{8}\sigma\upsilon\nu(-12^0)+\frac{3}{8}+\frac{1}{2}\sigma\upsilon\nu 30^0+\frac{1}{8}\sigma\upsilon\nu 60^0+\frac{3}{8}+\frac{1}{2}\sigma\upsilon\nu 66^0+\frac{1}{8}\sigma\upsilon\nu 132^0$$

$$= \frac{9}{8} + \frac{1}{2}\Big[\underbrace{\sigma\upsilon\nu(-6^0) + \sigma\upsilon\nu 30^0 + \sigma\upsilon\nu 66^0}_{S_1}\Big] + \frac{1}{8}\Big[\underbrace{\sigma\upsilon\nu(-12^0) + \sigma\upsilon\nu 60^0 + \sigma\upsilon\nu 132^0}_{S_2}\Big]$$

$$= \frac{9}{8} + \frac{1}{2}S_1 + \frac{1}{8}S_2 \quad (4)$$

$$S_1 = \sigma\upsilon\nu(-6^0) + \sigma\upsilon\nu(-6^0+36^0) + \sigma\upsilon\nu(-6^0+2\cdot 36^0) \underset{\substack{\alpha=-6^0\\ \omega=36^0}}{\overset{(2)}{=}} \frac{\sigma\upsilon\nu 30^0\cdot\eta\mu(3\cdot 18^0)}{\eta\mu 18^0}$$

$$\overset{(3)}{=} \frac{\sigma\upsilon\nu 30^0\cdot\cancel{\eta\mu 18^0}\cdot(3-4\eta\mu^2 18^0)}{\cancel{\eta\mu 18^0}} = \sigma\upsilon\nu 30^0\cdot(3-4\eta\mu^2 18^0) = \frac{\sqrt{3}}{2}\cdot\left[3-4\left(\frac{\sqrt{5}-1}{4}\right)^2\right]$$

$$= \frac{\sqrt{3}}{2}\cdot\left(3-4\cdot\frac{3-\sqrt{5}}{8}\right) = \frac{\sqrt{3}}{2}\cdot\left(3-\frac{3-\sqrt{5}}{2}\right) = \frac{\sqrt{3}}{2}\cdot\frac{3+\sqrt{5}}{2} = \frac{3\sqrt{3}+\sqrt{15}}{4} \quad (5)$$

$$S_2 = \sigma\upsilon\nu(-12^0) + \sigma\upsilon\nu(-12^0+72^0) + \sigma\upsilon\nu(-12^0+2\cdot 72^0) \underset{\substack{\alpha=-12^0\\ \omega=72^0}}{\overset{(2)}{=}} \frac{\sigma\upsilon\nu 60^0\cdot\eta\mu(3\cdot 36^0)}{\eta\mu 36^0}$$

$$\overset{(3)}{=} \frac{\sigma\upsilon\nu 60^0\cdot\cancel{\eta\mu 36^0}\cdot(3-4\eta\mu^2 36^0)}{\cancel{\eta\mu 36^0}} = \sigma\upsilon\nu 60^0\cdot(3-4\eta\mu^2 36^0) = \frac{1}{2}\cdot(3-16\eta\mu^2 18^0\sigma\upsilon\nu^2 18^0)$$

$$= \frac{1}{2}\cdot\left[3-16\frac{3-\sqrt{5}}{8}(1-\eta\mu^2 18^0)\right] = \frac{1}{2}\cdot\left[3-2(3-\sqrt{5})\left(1-\frac{3-\sqrt{5}}{8}\right)\right] = \frac{1}{2}\cdot\left[3-2(3-\sqrt{5})\frac{5+\sqrt{5}}{8}\right]$$

$$= \frac{1}{2}\cdot\left(3-\frac{10-2\sqrt{5}}{4}\right) = \frac{1}{2}\cdot\left(3-\frac{5-\sqrt{5}}{2}\right) = \frac{1}{2}\cdot\frac{1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{4} \quad (6)$$

$$(4) \underset{(6)}{\overset{(5)}{\Rightarrow}} S = \frac{9}{8} + \frac{1}{2}\cdot\frac{3\sqrt{3}+\sqrt{15}}{4} + \frac{1}{8}\cdot\frac{1+\sqrt{5}}{4} \Rightarrow \boxed{S = \frac{37+12\sqrt{3}+4\sqrt{15}+\sqrt{5}}{32}}$$

Ζήτημα 2ο

1η λύση

$$\beta\,\text{συν}A\,\text{συν}\Gamma + \gamma(\text{συν}A\,\text{συν}B - \text{συν}\Gamma) = \beta\cdot\text{συν}A\cdot\text{συν}\Gamma + \gamma\cdot\text{συν}A\cdot\text{συν}B - \gamma\cdot\text{συν}\Gamma$$

$$= \text{συν}A\cdot(\beta\cdot\text{συν}\Gamma + \gamma\cdot\text{συν}B) - \gamma\cdot\text{συν}\Gamma$$

$$= \text{συν}A\cdot(2R\cdot\text{ημ}B\cdot\text{συν}\Gamma + 2R\cdot\text{ημ}\Gamma\cdot\text{συν}B) - \gamma\cdot\text{συν}\Gamma$$

$$= 2R\cdot\text{συν}A\cdot(\text{ημ}B\cdot\text{συν}\Gamma + \text{ημ}\Gamma\cdot\text{συν}B) - \gamma\cdot\text{συν}\Gamma$$

$$= 2R\cdot\text{συν}A\cdot\text{ημ}(B+\Gamma) - \gamma\cdot\text{συν}\Gamma$$

$$= 2R\cdot\text{ημ}A\cdot\text{συν}A - \gamma\cdot\text{συν}\Gamma = \alpha\cdot\text{συν}A - \gamma\cdot\text{συν}\Gamma$$

$$= \alpha\cdot\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma} - \gamma\cdot\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}$$

$$= \frac{\alpha^2\beta^2+\alpha^2\gamma^2-\alpha^4}{2\alpha\beta\gamma} - \frac{\alpha^2\gamma^2+\beta^2\gamma^2-\gamma^4}{2\alpha\beta\gamma}$$

$$= \frac{\gamma^4-\alpha^4-\beta^2\gamma^2+\alpha^2\beta^2}{2\alpha\beta\gamma}$$

$$= \frac{(\gamma^2-\alpha^2)(\gamma^2+\alpha^2)-\beta^2(\gamma^2-\alpha^2)}{2\alpha\beta\gamma}$$

$$= \frac{(\gamma^2-\alpha^2)(\gamma^2+\alpha^2-\beta^2)}{2\alpha\beta\gamma} = \frac{\gamma^2-\alpha^2}{\beta}\cdot\frac{\gamma^2+\alpha^2-\beta^2}{2\alpha\gamma}$$

$$= \frac{\gamma^2-\alpha^2}{\beta}\cdot\text{συν}B$$

2η λύση

$$\frac{\gamma^2-\alpha^2}{\beta}\cdot\text{συν}B = \frac{4R^2\text{ημ}^2\Gamma-4R^2\text{ημ}^2A}{2R\text{ημ}B}\cdot\text{συν}B = 2R\cdot\frac{\text{ημ}^2\Gamma-\text{ημ}^2A}{\text{ημ}B}\cdot\text{συν}B$$

$$= 2R\cdot\frac{(\text{ημ}\Gamma-\text{ημ}A)\cdot(\text{ημ}\Gamma+\text{ημ}A)}{\text{ημ}B}\cdot\text{συν}B$$

$$= 2R\cdot\frac{2\text{ημ}\frac{\Gamma-A}{2}\text{συν}\frac{\Gamma+A}{2}\cdot 2\text{ημ}\frac{\Gamma+A}{2}\text{συν}\frac{\Gamma-A}{2}}{\text{ημ}B}\cdot\text{συν}B$$

$$= 2R\cdot\frac{\text{ημ}(\Gamma-A)\cdot\cancel{\text{ημ}(\Gamma+A)}}{\cancel{\text{ημ}B}}\cdot\text{συν}B = 2R\cdot\text{ημ}(\Gamma-A)\cdot\text{συν}B$$

$$= 2R\cdot\text{ημ}(A-\Gamma)\cdot\text{συν}(A+\Gamma)$$

$$= 2R\cdot(\text{ημ}A\text{συν}\Gamma-\text{ημ}\Gamma\text{συν}A)\cdot(\text{συν}A\text{συν}\Gamma-\text{ημ}A\text{ημ}\Gamma)$$

$$= 2R(\text{ημ}A\text{συν}A\text{συν}^2\Gamma-\text{ημ}^2A\text{ημ}\Gamma\text{συν}\Gamma-\text{συν}^2A\text{ημ}\Gamma\text{συν}\Gamma+\text{ημ}A\text{συν}A\text{ημ}^2\Gamma)$$

$$= 2R\cdot(\text{ημ}A\text{συν}A-\text{ημ}\Gamma\text{συν}\Gamma) = 2R\cdot\text{συν}A\cdot\text{ημ}(B+\Gamma) - 2R\text{ημ}\Gamma\text{συν}\Gamma$$

$$= 2R\cdot\text{συν}A\cdot(\text{ημ}B\text{συν}\Gamma+\text{ημ}\Gamma\text{συν}B) - \gamma\text{συν}\Gamma$$

$$= 2R\text{ημ}B\text{συν}A\text{συν}\Gamma + 2R\text{ημ}\Gamma\text{συν}A\text{συν}B - \gamma\text{συν}\Gamma$$

$$= \beta\text{συν}A\text{συν}\Gamma + \gamma\text{συν}A\text{συν}B - \gamma\text{συν}\Gamma$$

$$= \beta\text{συν}A\text{συν}\Gamma + \gamma(\text{συν}A\text{συν}B - \text{συν}\Gamma)$$

Ζήτημα 3ο

$$
\begin{aligned}
S &= \sum\left[\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot(1 - \sigma\upsilon\nu(\theta_\kappa - \theta_\lambda))\right] \\
&= \sum\left[\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2} - \alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot\sigma\upsilon\nu(\theta_\kappa - \theta_\lambda)\right] \\
&= \sum\left(\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\right) - \sum\left[\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot\sigma\upsilon\nu(\theta_\kappa - \theta_\lambda)\right] \\
&= \sum\alpha_\kappa^{\,2}\cdot\sum\alpha_\lambda^{\,2} - \sum\left[\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot(\sigma\upsilon\nu\theta_\kappa\cdot\sigma\upsilon\nu\theta_\lambda + \eta\mu\theta_\kappa\cdot\eta\mu\theta_\lambda)\right] \\
&= 1\cdot 1 - \sum\left(\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot\sigma\upsilon\nu\theta_\kappa\cdot\sigma\upsilon\nu\theta_\lambda + \alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot\eta\mu\theta_\kappa\cdot\eta\mu\theta_\lambda\right) \\
&= 1 - \left[\sum\left(\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot\sigma\upsilon\nu\theta_\kappa\cdot\sigma\upsilon\nu\theta_\lambda\right) + \sum\left(\alpha_\kappa^{\,2}\cdot\alpha_\lambda^{\,2}\cdot\eta\mu\theta_\kappa\cdot\eta\mu\theta_\lambda\right)\right] \\
&= 1 - \left[\sum\left[\left(\alpha_\kappa^{\,2}\cdot\sigma\upsilon\nu\theta_\kappa\right)\cdot\left(\alpha_\lambda^{\,2}\cdot\sigma\upsilon\nu\theta_\lambda\right)\right] + \sum\left[\left(\alpha_\kappa^{\,2}\cdot\eta\mu\theta_\kappa\right)\cdot\left(\alpha_\lambda^{\,2}\cdot\eta\mu\theta_\lambda\right)\right]\right] \\
&= 1 - \left[\sum\left(\alpha_\kappa^{\,2}\cdot\sigma\upsilon\nu\theta_\kappa\right)\cdot\sum\left(\alpha_\lambda^{\,2}\cdot\sigma\upsilon\nu\theta_\lambda\right) + \sum\left(\alpha_\kappa^{\,2}\cdot\eta\mu\theta_\kappa\right)\cdot\sum\left(\alpha_\lambda^{\,2}\cdot\eta\mu\theta_\lambda\right)\right] \\
&= 1 - \left[\left[\sum\left(\alpha_\kappa^{\,2}\cdot\sigma\upsilon\nu\theta_\kappa\right)\right]^2 + \left[\sum\left(\alpha_\kappa^{\,2}\cdot\eta\mu\theta_\kappa\right)\right]^2\right]
\end{aligned}
$$

Είναι $S \le 1$, διότι $\left[\sum\left(\alpha_\kappa^{\,2}\sigma\upsilon\nu\theta_\kappa\right)\right]^2 \ge 0$ και $\left[\sum\left(\alpha_\kappa^{\,2}\eta\mu\theta_\kappa\right)\right]^2 \ge 0$.