

ΓΕΝΙΚΕΣ ΕΙΣΙΤΗΡΙΕΣ ΕΞΕΤΑΣΕΙΣ 1970

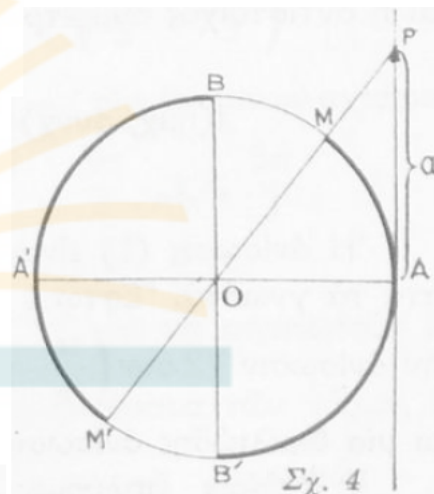
ΑΠΑΝΤΗΣΕΙΣ ΣΤΗΝ ΤΡΙΓΩΝΟΜΕΤΡΙΑ

ΠΟΛΥΤΕΧΝΙΚΟΣ ΚΥΚΛΟΣ

Σάββατο 5 Σεπτεμβρίου 1970

Ζήτημα 1°

$\varepsilon\varphi\chi < \alpha$. Ἡ ἀνίσωσις αὕτη ἔχει πάντοτε λύσιν, ἐφ' ὅσον $\alpha \in \mathbb{R}$, τὴν ὁποίαν εὐρίσκομεν ὡς ἑξῆς: Ἐστω $\alpha > 0$ (ἐὰν $\alpha < 0$ ἐργαζόμεθα ἀναλόγως). Ἐπὶ τοῦ ἄξονος τῶν ἐφαπτομένων λαμβάνομεν διάνυσμα \overline{AP} τοιοῦτον, ὥστε $(\overline{AP}) = \alpha$ καὶ θεωροῦμεν τὴν εὐθεῖαν τὴν διερχομένην ἐκ τῶν σημείων O καὶ P , τέμνουσαν τὴν περιφέρειαν εἰς τὰ M καὶ M' (Σχ. 4). Εἶναι ἤδη προφανές ἐκ τοῦ σχήματος, ὅτι κάθε τόξον, τὸ ὁποῖον ἔχει πέρασ τυ-



χὸν σημεῖον τοῦ τόξου $\widehat{MAB'}$ ἢ τοῦ τόξου $\widehat{BA'M'}$ (ἐξαιρουμένων τῶν ἄκρων M καὶ B' ἢ B καὶ M') ἐπαληθεύει τὴν ἀνίσωσιν.

Ἐν συνεχείᾳ, ἔστωσαν φ καὶ $\pi + \varphi$ τὰ μοναδικὰ τόξα τοῦ διαστήματος $[0, 2\pi]$ μὲ $\varepsilon\varphi\varphi = \varepsilon\varphi(\pi + \varphi) = \alpha$ ($0 < \varphi < \frac{\pi}{2}$). Τότε ἡ εἰδικὴ λύσις εἶναι:

$$\left(\frac{\pi}{2}, \pi + \varphi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right] \cup [0, \varphi).$$

Ἡ γενικὴ λύσις ἐν προκειμένῳ εὐρίσκεται ταχύτερον, ἀρκεῖ εἰς τὰ ἄκρα τοῦ διαστήματος $\left(\frac{\pi}{2}, \pi + \varphi\right)$ νὰ προσθέσωμεν τὸ $k\pi$ μὲ $k \in \mathbb{Z}$ (τυχόν). Ἦτοι, ἐὰν

$\Delta_k = \left(k\pi + \frac{\pi}{2}, k\pi + \pi + \varphi\right)$, αὕτη εἶναι:

$$\bigcup_{k \in \mathbb{Z}} \Delta_k = \left\{ \chi \in \mathbb{R} : \chi = k\pi + \theta, k \in \mathbb{Z}, \frac{\pi}{2} < \theta < \pi + \varphi \right\}.$$

Ζήτημα 2°

1^η λύση

$$\tau \cdot \sigma\upsilon\nu(B - \Gamma) \leq \upsilon \cdot (1 + \sqrt{2}) \Leftrightarrow$$

$$\frac{\alpha + \beta + \gamma}{2} \cdot (\sigma\upsilon\nu B \cdot \sigma\upsilon\nu \Gamma + \eta\mu B \cdot \eta\mu \Gamma) \leq \upsilon \cdot (1 + \sqrt{2}) \Leftrightarrow$$

$$\frac{\alpha + \beta + \gamma}{2} \cdot \left(\frac{\gamma \cdot \beta}{\alpha \cdot \alpha} + \frac{\beta \cdot \gamma}{\alpha \cdot \alpha} \right) \leq \upsilon \cdot (1 + \sqrt{2}) \Leftrightarrow$$

$$\frac{\alpha + \beta + \gamma}{2} \cdot \frac{2\beta\gamma}{\alpha^2} \leq \frac{\beta\gamma}{\alpha} \cdot (1 + \sqrt{2}) \Leftrightarrow$$

$$\frac{\alpha + \beta + \gamma}{\alpha} \leq 1 + \sqrt{2} \Leftrightarrow$$

$$1 + \frac{\beta + \gamma}{\alpha} \leq 1 + \sqrt{2} \Leftrightarrow$$

$$\frac{\beta + \gamma}{\alpha} \leq \sqrt{2} \Leftrightarrow$$

$$\beta + \gamma \leq \alpha\sqrt{2} \Leftrightarrow$$

$$(\beta + \gamma)^2 \leq 2\alpha^2 \Leftrightarrow$$

$$\beta^2 + 2\beta\gamma + \gamma^2 \leq 2(\beta^2 + \gamma^2) \Leftrightarrow$$

$$\beta^2 + 2\beta\gamma + \gamma^2 \leq 2\beta^2 + 2\gamma^2 \Leftrightarrow$$

$$0 \leq \beta^2 - 2\beta\gamma + \gamma^2 \Leftrightarrow$$

$$0 \leq (\beta - \gamma)^2 \text{ που ισχύει}$$

2^η λύση

Αν ΑΕ, ΑΔ είναι η διάμεσος και το ύψος

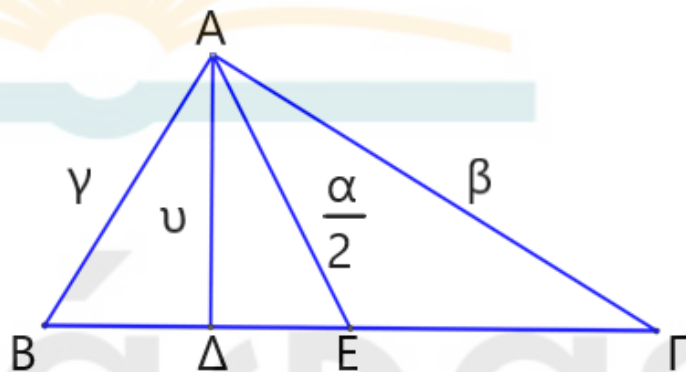
αντίστοιχα του ορθογωνίου τριγώνου, τότε $\hat{\Delta\hat{A}E} = |B - \Gamma|$.

$$\sigma\upsilon\nu(B - \Gamma) = \sigma\upsilon\nu \hat{\Delta\hat{A}E} = \frac{A\Delta}{A\hat{E}} = \frac{\upsilon}{\frac{\alpha}{2}} = \frac{2\upsilon}{\alpha} \Rightarrow$$

$$\begin{aligned} \tau \cdot \sigma\upsilon\nu(B - \Gamma) &= \tau \cdot \frac{2\upsilon}{\alpha} = \upsilon \cdot \frac{2\tau}{\alpha} = \upsilon \cdot \frac{\alpha + \beta + \gamma}{\alpha} = \upsilon \cdot \left(1 + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} \right) = \upsilon \cdot (1 + \eta\mu B + \eta\mu \Gamma) \\ &= \upsilon \cdot \left(1 + 2 \cdot \eta\mu \frac{B + \Gamma}{2} \cdot \sigma\upsilon\nu \frac{B - \Gamma}{2} \right) = \upsilon \cdot \left(1 + \chi \cdot \frac{\sqrt{2}}{\chi} \cdot \sigma\upsilon\nu \frac{B - \Gamma}{2} \right) \end{aligned}$$

$$\text{Είναι } \sigma\upsilon\nu \frac{B - \Gamma}{2} \leq 1, \text{ άρα } 1 + \sqrt{2} \cdot \sigma\upsilon\nu \frac{B - \Gamma}{2} \leq 1 + \sqrt{2}$$

$$\text{Επομένως } \tau \cdot \sigma\upsilon\nu(B - \Gamma) = \upsilon \cdot \left(1 + \sqrt{2} \cdot \sigma\upsilon\nu \frac{B - \Gamma}{2} \right) \leq \upsilon \cdot (1 + \sqrt{2})$$



Ζήτημα 3^ο

1^η λύση

$$\begin{aligned}\alpha^3 \cdot \sigma\upsilon\nu(B - \Gamma) &= (2R \cdot \eta\mu A)^3 \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 8R^3 \cdot \eta\mu^3 A \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 4R^3 \cdot \eta\mu^2 A \cdot 2\eta\mu A \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 4R^3 \cdot \frac{1 - \sigma\upsilon\nu 2A}{2} \cdot 2\eta\mu(B + \Gamma) \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 2R^3 \cdot (1 - \sigma\upsilon\nu 2A) \cdot (\eta\mu 2B + \eta\mu 2\Gamma) \\ &= 2R^3 \cdot (\eta\mu 2B + \eta\mu 2\Gamma - \eta\mu 2B \cdot \sigma\upsilon\nu 2A - \eta\mu 2\Gamma \cdot \sigma\upsilon\nu 2A)\end{aligned}$$

$$\acute{\alpha}\rho\alpha \alpha^3 \cdot \sigma\upsilon\nu(B - \Gamma) = 2R^3 \cdot (\eta\mu 2B + \eta\mu 2\Gamma - \eta\mu 2B \cdot \sigma\upsilon\nu 2A - \eta\mu 2\Gamma \cdot \sigma\upsilon\nu 2A) \quad (1)$$

Όμοια αποδεικνύουμε ότι :

$$\beta^3 \cdot \sigma\upsilon\nu(\Gamma - A) = 2R^3 \cdot (\eta\mu 2\Gamma + \eta\mu 2A - \eta\mu 2\Gamma \cdot \sigma\upsilon\nu 2B - \eta\mu 2A \cdot \sigma\upsilon\nu 2B) \quad (2)$$

$$\gamma^3 \cdot \sigma\upsilon\nu(A - B) = 2R^3 \cdot (\eta\mu 2A + \eta\mu 2B - \eta\mu 2A \cdot \sigma\upsilon\nu 2\Gamma - \eta\mu 2B \cdot \sigma\upsilon\nu 2\Gamma) \quad (3)$$

$$K = \alpha^3 \cdot \sigma\upsilon\nu(B - \Gamma) + \beta^3 \cdot \sigma\upsilon\nu(\Gamma - A) + \gamma^3 \cdot \sigma\upsilon\nu(A - B)$$

(1), (2)

$$\stackrel{(3)}{=} 2R^3 \cdot (2\eta\mu 2A + 2\eta\mu 2B + 2\eta\mu 2\Gamma - \eta\mu 2B \sigma\upsilon\nu 2A - \eta\mu 2\Gamma \sigma\upsilon\nu 2A - \eta\mu 2\Gamma \sigma\upsilon\nu 2B - \eta\mu 2A \sigma\upsilon\nu 2B - \eta\mu 2A \sigma\upsilon\nu 2\Gamma - \eta\mu 2B \sigma\upsilon\nu 2\Gamma)$$

$$= 2R^3 \cdot [2\eta\mu 2A + 2\eta\mu 2B + 2\eta\mu 2\Gamma - (\eta\mu 2A \sigma\upsilon\nu 2B + \eta\mu 2B \sigma\upsilon\nu 2A) - (\eta\mu 2B \sigma\upsilon\nu 2\Gamma + \eta\mu 2\Gamma \sigma\upsilon\nu 2B) - (\eta\mu 2\Gamma \sigma\upsilon\nu 2A + \eta\mu 2A \sigma\upsilon\nu 2\Gamma)]$$

$$= 2R^3 \cdot [2 \cdot \eta\mu 2A + 2 \cdot \eta\mu 2B + 2 \cdot \eta\mu 2\Gamma - \eta\mu(2A + 2B) - \eta\mu(2B + 2\Gamma) - \eta\mu(2\Gamma + 2A)]$$

$$= 2R^3 \cdot [2 \cdot \eta\mu 2A + 2 \cdot \eta\mu 2B + 2 \cdot \eta\mu 2\Gamma - \eta\mu(360^\circ - 2\Gamma) - \eta\mu(360^\circ - 2A) - \eta\mu(360^\circ - 2B)]$$

$$= 2R^3 \cdot (2 \cdot \eta\mu 2A + 2 \cdot \eta\mu 2B + 2 \cdot \eta\mu 2\Gamma + \eta\mu 2\Gamma + \eta\mu 2A + \eta\mu 2B)$$

$$= 2R^3 \cdot (3 \cdot \eta\mu 2A + 3 \cdot \eta\mu 2B + 3 \cdot \eta\mu 2\Gamma)$$

$$= 6R^3 \cdot (\eta\mu 2A + \eta\mu 2B + \eta\mu 2\Gamma)$$

$$= 6R^3 \cdot [2 \cdot \eta\mu A \cdot \sigma\upsilon\nu A + 2 \cdot \eta\mu(B + \Gamma) \cdot \sigma\upsilon\nu(B - \Gamma)]$$

$$= 6R^3 \cdot [2 \cdot \eta\mu A \cdot \sigma\upsilon\nu A + 2 \cdot \eta\mu A \cdot \sigma\upsilon\nu(B - \Gamma)]$$

$$= 6R^3 \cdot 2 \cdot \eta\mu A \cdot [\sigma\upsilon\nu A + \sigma\upsilon\nu(B - \Gamma)]$$

$$= 12R^3 \cdot \eta\mu A \cdot [\sigma\upsilon\nu(B + \Gamma) + \sigma\upsilon\nu(B - \Gamma)]$$

$$= 12R^3 \cdot \eta\mu A \cdot 2 \cdot \eta\mu B \cdot \eta\mu \Gamma$$

$$= 24R^3 \cdot \frac{\alpha}{2R} \cdot \frac{\beta}{2R} \cdot \frac{\gamma}{2R}$$

$$= 3\alpha\beta\gamma$$

$$= 12 \cdot \frac{\alpha\beta\gamma}{4R} \cdot R$$

$$= 12 \cdot E \cdot R$$

2^η λύση

$$\begin{aligned}\alpha^3 \cdot \sigma\upsilon\nu(B - \Gamma) &= (2R \cdot \eta\mu A)^3 \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 8R^3 \cdot \eta\mu^3 A \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 4R^3 \cdot \eta\mu^2 A \cdot 2\eta\mu A \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 4R^3 \cdot \eta\mu^2 A \cdot 2\eta\mu(B + \Gamma) \cdot \sigma\upsilon\nu(B - \Gamma) \\ &= 4R^3 \cdot \eta\mu^2 A \cdot (\eta\mu^2 B + \eta\mu^2 \Gamma) \\ &= 4R^3 \cdot (\eta\mu^2 A \cdot \eta\mu^2 B + \eta\mu^2 A \cdot \eta\mu^2 \Gamma)\end{aligned}$$

$$\acute{\alpha}\rho\alpha \alpha^3 \cdot \sigma\upsilon\nu(B - \Gamma) = 4R^3 \cdot (\eta\mu^2 A \cdot \eta\mu^2 B + \eta\mu^2 A \cdot \eta\mu^2 \Gamma) \quad (1)$$

Όμοια αποδεικνύουμε ότι :

$$\beta^3 \cdot \sigma\upsilon\nu(\Gamma - A) = 4R^3 \cdot (\eta\mu^2 B \cdot \eta\mu^2 \Gamma + \eta\mu^2 B \cdot \eta\mu^2 A) \quad (2)$$

$$\gamma^3 \cdot \sigma\upsilon\nu(A - B) = 4R^3 \cdot (\eta\mu^2 \Gamma \cdot \eta\mu^2 A + \eta\mu^2 \Gamma \cdot \eta\mu^2 B) \quad (3)$$

$$K = \alpha^3 \cdot \sigma\upsilon\nu(B - \Gamma) + \beta^3 \cdot \sigma\upsilon\nu(\Gamma - A) + \gamma^3 \cdot \sigma\upsilon\nu(A - B)$$

$$\begin{aligned}&\stackrel{(1), (2)}{=} 4R^3 \cdot (\eta\mu^2 A \cdot \eta\mu^2 B + \eta\mu^2 A \cdot \eta\mu^2 \Gamma + \eta\mu^2 B \cdot \eta\mu^2 \Gamma + \eta\mu^2 B \cdot \eta\mu^2 A + \eta\mu^2 \Gamma \cdot \eta\mu^2 A + \eta\mu^2 \Gamma \cdot \eta\mu^2 B) \\ &\stackrel{(3)}{=} 4R^3 [(\eta\mu^2 A \eta\mu^2 B + \eta\mu^2 B \eta\mu^2 A) + (\eta\mu^2 B \eta\mu^2 \Gamma + \eta\mu^2 \Gamma \eta\mu^2 B) + (\eta\mu^2 A \eta\mu^2 \Gamma + \eta\mu^2 \Gamma \eta\mu^2 A)] \\ &= 4R^3 [(2\eta\mu^2 A \eta\mu B \sigma\upsilon\nu B + 2\eta\mu^2 B \eta\mu A \sigma\upsilon\nu A) + (2\eta\mu^2 B \eta\mu \Gamma \sigma\upsilon\nu \Gamma + 2\eta\mu^2 \Gamma \eta\mu B \sigma\upsilon\nu B) + (2\eta\mu^2 A \eta\mu \Gamma \sigma\upsilon\nu \Gamma + 2\eta\mu^2 \Gamma \eta\mu A \sigma\upsilon\nu A)] \\ &= 4R^3 [2\eta\mu A \eta\mu B (\eta\mu A \sigma\upsilon\nu B + \eta\mu B \sigma\upsilon\nu A) + 2\eta\mu B \eta\mu \Gamma (\eta\mu B \sigma\upsilon\nu \Gamma + \eta\mu \Gamma \sigma\upsilon\nu B) + 2\eta\mu A \eta\mu \Gamma (\eta\mu A \sigma\upsilon\nu \Gamma + \eta\mu \Gamma \sigma\upsilon\nu A)] \\ &= 4R^3 \cdot [2 \cdot \eta\mu A \cdot \eta\mu B \cdot \eta\mu (A + B) + 2 \cdot \eta\mu B \cdot \eta\mu \Gamma \cdot \eta\mu (B + \Gamma) + 2 \cdot \eta\mu A \cdot \eta\mu \Gamma \cdot \eta\mu (A + \Gamma)] \\ &= 4R^3 \cdot (2 \cdot \eta\mu A \cdot \eta\mu B \cdot \eta\mu \Gamma + 2 \cdot \eta\mu B \cdot \eta\mu \Gamma \cdot \eta\mu A + 2 \cdot \eta\mu A \cdot \eta\mu \Gamma \cdot \eta\mu B) \\ &= 4R^3 \cdot 6 \cdot \eta\mu A \cdot \eta\mu B \cdot \eta\mu \Gamma \\ &= 24R^3 \cdot \frac{\alpha}{2R} \cdot \frac{\beta}{2R} \cdot \frac{\gamma}{2R} \\ &= 3\alpha\beta\gamma \\ &= 12 \cdot \frac{\alpha\beta\gamma}{4R} \cdot R \\ &= 12 \cdot E \cdot R\end{aligned}$$